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Existence and Regularity of Extrema

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Consider the minimization of a possibly noncoercive Gâteaux differentiable functional $F: \mathfrak{X} \to \bar{\mathbb{R}}$. A modified notion of coercivity is introduced which may be usable to show existence of a minimum. Alternatively, if $\hat{F}: \mathfrak{Y} \to \bar{\mathbb{R}}$ has a minimum at $\bar{y} \in \mathfrak{Y}$ ($\hat{F}$ not differentiable but the restriction F of $\hat{F}$ to $\mathfrak{X} \subset \mathfrak{Y}$ differentiable), one may be able to show $\bar{y}$ is actually in $\mathfrak{X}$. The latter case is related to justification of formally calculated "necessary conditions" for optimal controls. The arguments are applications of Ekeland's "approximate variational principle" (*J. Math. Anal. Appl.* **47** (1974), 324–353).

1. INTRODUCTION

For motivation, let us note a typical difficulty arising in optimal control theory, especially for distributed parameter problems. We may find the following situation:

(a) We can obtain coercivity of the cost functional $\mathcal{F}$ on a space $\mathfrak{Y}$ and show existence of an optimal control u in $\mathfrak{Y}$;

(b) We can *formally* determine "necessary conditions" for u (corresponding to setting $\mathcal{F}'(u) = 0$ under the assumption of differentiability);

(c) We can justify these "necessary conditions" under the assumption that u is in a space $\mathfrak{X}$ ($\mathfrak{X} \subset \mathfrak{Y}$);

(d) We can show that u is in $\mathfrak{X}$ under the assumption that it satisfies the "necessary conditions."

Clearly (c) and (d) correspond to a circular argument which cannot quite be resolved by (a). One resolution would be to strengthen the argument of (a) to show that $\mathcal{F}$ attains its minimum in the more restricted space $\mathfrak{X}$, i.e., *existence* of an optimal control $\hat{u}$ in $\mathfrak{X}$—which, by (c), will satisfy the

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necessary conditions. Lacking a uniqueness result in $\mathfrak{Y}$ for the optimal control, this leaves open the possibility that a (different) optimal control $\hat{u}$ might exist in $\mathfrak{Y}$ which did not satisfy the necessary conditions (if it did, then we would have $\hat{u}$ in $\mathfrak{X}$ by (d)). We therefore wish a *regularity* result—that an optimal control $\hat{u}$ in $\mathfrak{Y}$ must necessarily be in $\mathfrak{X}$. With this situation in mind, we proceed to the general setting.

X be a Banach space and $F\colon \mathfrak{X} \to \bar{\mathbb{R}} := (-, +\infty)$ a semibounded functional, not identically $+\infty$. We then have

$$-\infty < F_* := \inf F < \infty, \qquad F_* \leqslant F(x) \leqslant \infty \quad \text{for} \quad x \in \mathfrak{X}.$$

We set

$$\mathscr{S}_a := \{x\colon F(x) \leqslant a\}, \qquad \mathscr{S}_* := \mathscr{S}_{F_*} = \{x\colon F(x) \leqslant F(x')\ \forall x' \in \mathfrak{X}\}$$

and note that minimizing F is equivalent to minimizing F over $\mathscr{S}_a$ (any $a \geqslant F_*$) while for F to attain its minimum just means that $\mathscr{S}_*$ is not empty.

The functional F is called *coercive* if each $\mathscr{S}_a$ is bounded. This is of interest for the so-called *direct method of the calculus of variations.*

THEOREM 1.1. *Let $\mathfrak{X}$ be reflexive and F lower semicontinuous* (*lsc*) *from $\mathfrak{X}_w := [\mathfrak{X}$ with its weak topology$]$. Then if F is coercive, it attains its minimum.* ∎

For many variational problems the principal difficulty is proving coercivity. With this in mind we introduce a modified coercivity condition involving the Gâteaux derivative F'.

DEFINITION 1.2. We say that $F\colon X \to \bar{\mathbb{R}}$ is *F'-coercive* if, for some $a > F_*$, the Gâteaux deriative F' exists on $\mathscr{S}_a$ (so the set

$$\mathscr{S}_{a,\varepsilon} := \{x\colon F(x) \leqslant a, \|F'(x)\| \leqslant \varepsilon\}$$

is defined) and if, for some $\varepsilon > 0$ and some bound M, we have

$$\inf\{F(x)\colon x \in \mathscr{S}_{a,\varepsilon}\} = \inf\{F(x)\colon \|x\| \leqslant M\}.$$

(Clearly, a sufficient condition for this is that F be *strictly F'-coercive*: $\mathscr{S}_{a,\varepsilon}$ is defined and bounded for some $a > F_*$, $\varepsilon > 0$. We shall note that strict F'-coercivity is almost equivalent—given differentiability—to ordinary coercivity of a modified functional with the same minimum.) Our first principal result will be

THEOREM 1.3. *Let $\mathfrak{X}$ be reflexive and let $F\colon \mathfrak{X} \to \bar{\mathbb{R}}$ be lower semicontinuous from $\mathfrak{X}_w$. Then if F is F'-coercive it attains its minimum.*

The example $F(x)=x^2e^{-x}$ shows that, even in the one-dimensional case $\mathfrak{Y}=\mathbb{R}$, one can find F which is F'-coercive but not coercive. Thus Theorem 1.3 is an improvement on the "standard" Theorem 1.1. This will be a *useful* improvement to the extent that we may, on occasion, find it easier to verify F'-coercivity (or even strict F'-coercivity) than to verify ordinary coercivity.

Next, suppose $\mathfrak{X}$ is embedded in a larger Banach space $\mathfrak{Y}$ so that F is the restriction to $\mathfrak{X}$ of $\hat{F}:\mathfrak{Y}\to\bar{\mathbb{R}}$. We assume it is known that $\hat{F}$ attains its minimum $\hat{F}_*$ at $\bar{y}$ in $\mathfrak{Y}$ and seek to obtain (the regularity result) that $\bar{y}$ must actually be in $\mathfrak{X}$. We envisage, in particular, a situation in which we have a function

$$\hat{G}:\mathfrak{Y}\times\mathfrak{Y}\to\mathfrak{X}^* \qquad \text{with} \quad F'(x)=\hat{G}(x,x) \quad \text{for} \quad x\in\mathscr{S}_a\subset\mathfrak{X} \tag{1.1}$$

satisfying

$$\hat{G}(x,y)=0, \qquad \hat{F}(y)\leqslant a \qquad \text{implies} \quad x\in\mathscr{B}\subset\mathfrak{X}. \tag{1.2}$$

If $\hat{F}$ were Gâteaux differentiable with $\hat{F}'(y)=\hat{G}(y,y)$—at $y=\bar{y}$, specifically—then, knowing that $\hat{F}'(\bar{y})=0$ for a minimizer $\bar{y}$, we could conclude from (1.2) that $\bar{y}$ is in $\mathfrak{X}$. If, however, we did *not* know Gâteaux differentiability for $\hat{F}$, we could not, in general, use (1.2) to conclude that $\bar{y}$ is in $\mathscr{B}\subset\mathfrak{X}$. We might have, in the sense of a generalized gradient, that $0\in\partial\hat{F}(\bar{y})$ but (without some convexity—as in [2, V. 3.5]) this need not be adequate to obtain regularity of $\bar{y}$. This difficulty corresponds directly to the circular argument (c) and (d): If $\hat{G}(\bar{y},\bar{y})=0$, then $\bar{y}\in\mathfrak{X}$ by (1.2); if $\bar{y}\in\mathfrak{X}$, then we can justify differentiability; if we have existence of F' at the minimizer $\bar{y}$, then $G(\bar{y},\bar{y})=F'(\bar{y})=0,\ldots$.

Our second principal result, whose precise statement is left to Section 3, is that (under suitable hypotheses regarding G) we can break the circle. What is needed is a form of F'-coercivity sufficiently selective to ensure not only that F attains its minimum (in $\mathfrak{X}$) but that this occurs at $\bar{y}$ so $\bar{y}$ is in $\mathfrak{X}$. (We shall construct a sequence weakly convergent in $\mathfrak{X}$ and converging to $\bar{y}$ in $\mathfrak{Y}$.)

The tool for both results is the "approximate variational principle" of Ekeland [1, Theorem 2.2]).

THEOREM 1.6 (Ekeland). *Let $F:\mathfrak{X}\to\bar{\mathbb{R}}$ be a Gâteaux differentiable lsc functional with $-\infty<F_*<\infty$. Then, for every $\varepsilon>0$, $\lambda>0$ and every $x\in\mathfrak{X}$ with $F(x)\leqslant F_*+\varepsilon$, there exists $\hat{x}\in\mathfrak{X}$ such that,*

$$F(\hat{x})\leqslant F(x), \qquad \|\hat{x}-x\|<\lambda, \qquad \|F'(\hat{x})\|\leqslant\varepsilon/\lambda. \quad \blacksquare$$

This is applied in [1, 2] to show that certain nonlinear maps have dense ranges but the present application seems new.

2. Existence

In this section we prove some general results to which Theorem 1.3 will be a corollary.

Definition 2.1. We shall say that a subset $\mathscr{B} \subset \mathfrak{X}$ is *variationally compact with respect to F* (briefly: $\mathscr{B}$ is F-vc) if: *for every sequence* $\{x_j\}$ *in* $\mathscr{B}$ *there exists* $\bar{x} \in \mathscr{B}$ *with* $F(\bar{x}) \leqslant \liminf F(x_j)$. We shall say $\mathscr{B}$ is *variationally precompact with respect to F* (briefly: $\mathscr{B}$ is F-vpc) if this holds with $\bar{x} \in \mathfrak{X}$ but not necessarily in $\mathscr{B}$.

We collect a number of obvious consequences,

Theorem 2.2. (a) *Every finite set (in particular, $\varnothing$) is F-vc. Every F-vc set is F-vpc.* (b) *If* $\inf_{\mathscr{B}} F > F_*$ *(more generally, if* $\inf_{\mathscr{B}} F \geqslant \inf_{\mathscr{C}} F$ *for some F-vpc set* $\mathscr{C}$*), then* $\mathscr{B}$ *is F-vpc.* (c) *If F attains its minimum* ($\mathscr{S}_* \neq \varnothing$)*, then every* $\mathscr{B} \subset \mathfrak{X}$ *is F-vpc. If* $\mathscr{B} \cap \mathscr{S}_* \neq \varnothing$*, then* $\mathscr{B}$ *is F-vc.* (d) *F attains its minimum if and only if there exists an F-vpc set* $\mathscr{B}$ *with* $\inf_{\mathscr{B}} F = F_*$. (e) *If* $\mathscr{B}$ *is compact with respect to a topology* τ *and F is lsc from* $\mathscr{B}_\tau := [\mathscr{B}$ *with the topology* $\tau]$*, then* $\mathscr{B}$ *is F-vc. In particular, if* $\mathfrak{X}$ *is reflexive and F is lsc from* $\mathfrak{X}_w$ *, then every bounded set is F-vpc and every* $\mathfrak{X}_w$*-closed bounded set is F-vc.* ∎

In view of Theorem 2.2(e), we note that Theorem 1.3 is a special case of

Theorem 2.3. *Let* $F\colon \mathfrak{X} \to \overline{\mathbb{R}}$ *with* $-\infty < F_* < \infty$ *and with F lsc and Gâteaux differentiable on* $\mathscr{S}_a := \{x\colon F(x) \leqslant a\}$ *for some* $a > F_*$. *Suppose there is an F-vpc set* $\mathscr{B} \subset \mathfrak{X}$ *such that, for some* $\varepsilon > 0$,

(∗) *for each* $x \mathscr{S}_{a,\varepsilon} := \{x \in \mathscr{S}_a\colon \|F'(x)\| \leqslant \varepsilon\}$, *there exists* $\hat{x} \in \mathscr{B}$ *with* $F(\hat{x}) \leqslant F(x)$.

Then F attains its minimum.

Proof. Condition (∗) requires that $\inf_{\mathscr{B}} F$ be no greater than the infimum over $\mathscr{S}_{a,\varepsilon}$. Thus, by Theorem 2.2(d), it is sufficient to show that $\mathscr{S}_{a,\varepsilon}$ contains a minimizing sequence. This, however, is an immediate consequence of Ekeland's theorem (1.6) since, for any minimizing sequence $\{x_j\}$ for F, we may take $\lambda_j = 1$ so $\hat{x}_j$ is in $\mathscr{S}_{a,\varepsilon}$. ∎

Proof of Theorem 1.3. As in the preceding proof we have $\inf\{F(x)\colon x \in \mathscr{S}_{a,\varepsilon}\} = F_*$ by Theorem 1.6. The hypothesis of F'-coercivity implies, then, that $\inf\{F(x)\colon \|x\| \leqslant \mu\} = F_*$ also, so there exists a bounded minimizing sequence for F and so, taking a subsequence, there exists a weakly convergent minimizing sequence. The lower semicontinuity from $\mathfrak{X}_w$ then implies that F attains F_* at the weak limit. ∎

The form of $\mathscr{S}_{a,\varepsilon}$ suggests considering a modified functional $\tilde{F}$ defined by

$$\tilde{F}(x) := F(x) + \|F'(x)\| \tag{2.1}$$

since, clearly, $\mathscr{S}_{a,\varepsilon} \subset \tilde{\mathscr{S}}_{a+\varepsilon} := \{: \tilde{F}(x) \leqslant a + \varepsilon\}$. (The precise form of (2.1) is unimportant—indeed, we might well wish to take $[F + \|F'\|^2]$ in a Hilbert space context.)

THEOREM 2.5. *Let $\mathfrak{X}$ be a reflexive and $F: \mathfrak{X} \to \bar{\mathbb{R}}$ with $-\infty < F_* < \infty$; suppose F is Gâteaux differentiable on $\mathscr{S}_a$ for some $a > F_*$ so* (2.1) *defines $\tilde{F}$ (at least on $\mathscr{S}_a$). Suppose F is lsc on $\mathscr{S}_a$ and $\tilde{F}$ is coercive (we need only that $\tilde{\mathscr{S}}_{\tilde{a}}$ is bounded for some $\tilde{a} > F_*$). Then F is F'-coercive and $F, \tilde{F}$ attain the same minimum F_* on the same set $\mathscr{S}_*$. (Note that no continuity condition is imposed on $\tilde{F}$.)*

Proof. As shown, one has $\inf\{F(x): x \in \mathscr{S}_{\bar{a},\varepsilon}\} = F_*$ by Theorem 1.6. We may take $F_* < \bar{a} < \bar{a} + \varepsilon < \tilde{a} \leqslant a$ so $\mathscr{S}_{\bar{a},\varepsilon} \subset \tilde{\mathscr{S}}_{\tilde{a}}$ which is bounded. Thus F is F'-coercive and so, by Theorem 1.3, attains its minimum ($\mathscr{S}_* \neq \varnothing$) since $F_* \leqslant F$, $F_* \leqslant \tilde{F}_*$. On the other hand, for x in $\mathscr{S}_*$ we have $F'(x) = 0$ so $\tilde{F}(x) = F(x) = F$ and $\tilde{F}$ also attains its minimum $\tilde{F}_* = F_*$ (precisely) on $\mathscr{S}_*$. ∎

Remark. Given the boundedness of $\tilde{\mathscr{S}}_{\tilde{a}}$ we easily see, using Theorem 1.6, that every minimizing sequence $\{x_j\}$ for F must be bounded. Let M be a bound on $\tilde{\mathscr{S}}_{\tilde{a}}$ and note that we have $\hat{x}_j$ such that $\|\hat{x}_j - x_j\| \leqslant 1$, $F(\hat{x}_j) \leqslant F(x_j) =: F_* + \varepsilon_j$ so $\tilde{F}(\hat{x}_j) \leqslant F_* + 2\varepsilon_j$ which is less than $\tilde{a}$ (for large j) so $\|x_j\| \leqslant M + 1$. Thus, the coercivity of $\tilde{F}$ *almost* implies coercivity of F. On the other hand, if F is coercive, then $\tilde{F}$ must certainly also be coercive. If F is F'-coercive, then we cannot conclude that $\tilde{F}$ is coercive (consider again the example $x^2 e^{-x}$) but can conclude that (for large enough M)

$$\begin{aligned} \tilde{\tilde{F}}(x) &:= \tilde{F}(x), \qquad \|x\| \leqslant M, \\ &:= +\infty, \qquad \|x\| > M, \end{aligned}$$

is coercive and attains its minimum on $\{x \in \mathscr{S}_* : \|x\| \leqslant M\}$.

A somewhat more delicate criterion than in Theorem 2.3 is given by

THEOREM 2.6. *Let $\mathfrak{X}$ be reflexive and F Gâteaux differentiable on $\mathscr{S}_a$ and semibounded and lsc from $\mathfrak{X}_w$. Suppose there is a number M and a function $\psi: (\mathbb{R}^+)^2 \to \mathbb{R}^+$ such that*

($*$) $\liminf_{\beta \to 0+} \psi(\alpha, \beta) < M$ *for each* $\alpha > 0$

and such that, given $\alpha, \beta > 0$, we have,

(**) *for any $\hat{x} \in \mathscr{S}_a$ with $\|F'(\hat{x})\| \leqslant \beta$ there exists $\tilde{x} \in \mathfrak{X}$ with*

$$F(\tilde{x}) \leqslant F(\hat{x}) + \alpha \qquad \text{and} \qquad \|\tilde{x}\| \leqslant \psi(\alpha, \beta).$$

Then F attains its minimum.

Proof. Given $\alpha > 0$, choose $\beta > 0$ such that $\psi(\alpha, \beta) \leqslant M$. Choose x so $F(x) \leqslant F_* + \alpha$ and, by Theorem 1.6 with $\varepsilon = \alpha$ and $\lambda = \alpha/\beta$, obtain $\hat{x}$ so $F(\hat{x}) \leqslant F(x) \leqslant F_* + \alpha$ and $\|F'(\hat{x})\| \leqslant \beta$. By (**) we have $\tilde{x}$ with $F(\tilde{x}) \leqslant F_* + 2\alpha$ and $\|\tilde{x}\| \leqslant \psi(\alpha, \beta) \leqslant M$. Taking a sequence $\alpha_j \to 0$, we have, correspondingly, a bounded minimizing sequence $\{\tilde{x}_j\}$ for F. By the lower semicontinuity from $\mathfrak{X}_w$, we have F_* attained at the weak limit of (a subsequence of) $\{\tilde{x}\}$. ∎

3. Regularity

We now assume $\mathfrak{X}$ is continuously embedded in another Banach space $\mathfrak{Y}$ and that F is the restriction to $\mathfrak{X}$ of $\hat{F}: \mathfrak{Y} \to \overline{\mathbb{R}}$. We begin by considering properties of a map $\hat{G}: \mathfrak{Y} \times \mathfrak{Y} \to \mathfrak{X}^*$ or, more precisely,

$$\hat{G}: \mathfrak{Y} \times \hat{\mathscr{S}}_a \to \mathfrak{X}^*, \qquad \hat{\mathscr{S}}_a := \{y: \hat{F}(y) \leqslant a\}.$$

We shall be concerned with $\hat{G}$ primarily for (y', y) such that $\hat{G}(y', y)$ is close to 0 in $\mathfrak{X}^*$. Our assumptions correspond to a weakened invertibility of $\hat{G}(\cdot, y)$ for such y.

Hypothesis 3.1. *Suppose $\{y_j\}$ is in $\hat{\mathscr{S}}_a \subset \mathfrak{Y}$ with $\hat{G}(y_j, y_j) \to 0$ in $\mathfrak{X}^*$ and $y_j \to \bar{y}$ in $\mathfrak{Y}$. If $\{x_j\}$ is in $\mathfrak{X}$ with $x_j \rightharpoonup \bar{x}$ (in $\mathfrak{X}_w$) and $\hat{G}(x_j, y_j) \to 0$ in $\mathfrak{X}^*$, then $\bar{x} = \bar{y}$.*

Hypothesis 3.2. *There are constants ε, $M > 0$, a function $\psi: \mathbb{R}^+ \to \overline{\mathbb{R}}$ with $\lim_{\beta \to 0+} \psi(\beta) = 0$, and a map $\mathbf{H}: \hat{\mathscr{S}}_a \to \mathfrak{X}$ such that:*

(i) $\|\mathbf{H}(y)\|_{\mathfrak{X}} \leqslant M$ *if* $\|\hat{G}(y, y)\| < \varepsilon$.

(ii) $\|\hat{G}(\mathbf{H}(y), y\| \leqslant \psi(\beta)$ *if* $y \in \hat{\mathscr{S}}_a$, $\|\hat{G}(y, y)\| \leqslant \beta$.

We shall later consider situations in which Hypotheses 3.1 and 3.2 can be verified. Meanwhile we note that (1.2) corresponds to $\mathbf{H}(\cdot)$ being determined implicitly by "solving" $\hat{G}(x, y) = 0$ for $x := \mathbf{H}(y)$.

Theorem 3.3. *Let $\hat{F}: \mathfrak{Y} \to \overline{\mathbb{R}}$ be a minimized at $\bar{y}$. Let $\mathfrak{X}$ be a reflexive Banach space continuously embedded in $\mathfrak{Y}$ in such a way that*

(*) *there is a sequence* $\{x_j\}$ *in* $\mathfrak{X}$ *such that* $x_j \to^{\mathfrak{Y}} \bar{y}$ *and* $F(x_j) \to \hat{F}(\bar{y}) = \hat{F}_*$ (*e.g.*, $\hat{F}$ *lsc and* $\mathfrak{X}$ *dense in* $\mathfrak{Y}$).

Assume that the restriction F *of* $\hat{F}$ *to* $\mathfrak{X}$ *is lsc and Gâteaux differentiable* (*in the sense of* $\mathfrak{X}$) *on* $\mathscr{S}_a := \{x \in \mathfrak{X}: F(x) \leqslant a\}$ *for* $a > F_* = \hat{F}_*$. *Suppose there is a function* $\hat{G}$ *satisfying* 1.1 *and Hypotheses* 3.1 *and* 3.2. *Then* $\bar{y}$ *is in* $\mathfrak{X}$ *with* $\|\bar{y}\|_{\mathfrak{X}} \leqslant M$.

Proof. We begin with the minimizing sequence $\{x_j\}$ in $\mathfrak{X}$ such that $x_j \to^{\mathfrak{Y}} \bar{y}$, as in (*). By Ekeland's theorem (1.6) there exists a new minimizing sequence $\{\hat{x}_j\}$ such that (taking $\varepsilon_j := [F(x_j) - F_*] \to 0$, $\lambda_j = \sqrt{\varepsilon_j}$) we have $F'(\hat{x}_j) \to 0$ in $\mathfrak{X}^*$ and $\|x_j - \hat{x}_j\|_{\mathfrak{X}} \leqslant \lambda_j \to 0$. From the latter, $\|x_j - \hat{x}_j\|_{\mathfrak{Y}} \to 0$ so, as $x_j \to^{\mathfrak{Y}} \bar{y}$, we also have $\hat{x}_j \to^{\mathfrak{Y}} \bar{y}$. We may assume, then, that $\{\hat{x}_j\}$ is in $\mathscr{S}_a$ and $\|\hat{G}(\hat{x}_j, \hat{x}_j)\| = \|F'(\hat{x}_j)\| \leqslant \varepsilon$. Now let $\tilde{x}_j := \mathbf{H}(\hat{x}_j)$.

By Hypothesis 3.2(i) we have $\|\tilde{x}_j\| \leqslant M$ so, extracting a subsequence if necessary, we have weak convergence (in $\mathfrak{X}_w$): $\tilde{x}_j \rightharpoonup \bar{x}$ for some $\bar{x}$ in $\mathfrak{X}$. On the other hand, $\hat{G}(\hat{x}_j, \hat{x}_j) \to 0$ so, as $\psi(0+) = 0$, Hypothesis 3.2(ii) gives $\hat{G}(\tilde{x}_j, \hat{x}_j) \to 0$ as well. Hypothesis 3.1 then ensures that, as $\tilde{x}_j \rightharpoonup \bar{x}$ in $\mathfrak{X}_w$ and $\hat{x}_j \to \bar{y}$ in $\mathfrak{Y}$, we have $\bar{x} = \bar{y}$. Thus, $\bar{y}$ is in $\mathfrak{X}$. Since $\|\tilde{x}_j\|_{\mathfrak{X}} \leqslant M$ for each j, we also have $\|\bar{y}\|_{\mathfrak{X}} \leqslant M$ in the limit. ∎

Let us now see how the structure of this result, with Hypotheses 3.1 and 3.2, can be related to the difficulty (a)–(d), described. We consider briefly, for example, a problem of the sort [4] which suggested these considerations.

Take $\mathfrak{Y} := L^2(\mathscr{Q})$ with $\mathscr{Q} := (0, T) \times \Omega$ (Ω bounded in $\mathbb{R}^m$ with $\partial\Omega$ smooth) and seek to minimize, say,

$$\hat{F}(y) := \tfrac{1}{2}\|y\|^2 + \tfrac{1}{2}\|z(T) - \zeta\|_\Omega^2$$

(ζ given on Ω) with $z = z[y]$ determined by

$$\dot{z} - \Delta z = f(\cdot, y, z, \nabla z), \qquad z|_{\partial\Omega} = 0, \quad z(0) = 0. \tag{3.1}$$

Under suitable assumptions on the smoothness and structure of $f = f(\omega, r, s, p)$ $[\omega \in Q; r, s \in \mathbb{R}; p \in \mathbb{R}^m]$ we can show (i) there is a unique solution $z = z[y]$ in $\mathfrak{X} := H^{2,1}(\mathscr{Q})$ (notation of [3]) for each $y \in \mathfrak{X}$ (so $\hat{F}(y)$ is finite for each $y \in \mathfrak{Y}$), (ii) the map $y \mapsto z[y]: \mathfrak{Y} \to \mathfrak{Z}$ is continuous and bounded, and (iii) there is an optimal control $\bar{y}$ in $\mathfrak{Y}$. Under a *formal assumption* of differentiability one sees that the Gâteaux derivative z_y of the map $y \mapsto z[y]$ would be given (formally differentiating (3.1) with respect to y) by $z_y: h \mapsto v$ with

$$\dot{v} - \Delta v - f_p \nabla v - f_s v = f_r h, \qquad v|_{\partial\Omega} = 0, \quad v(0) = 0$$

with f_r, f_s, f_p evaluated at $(\cdot, y, z, \nabla z)$. *Formally*, then, we have

$$\hat{F}'(y)\colon h \mapsto \langle y, h\rangle + \langle z(T) - \zeta, [z_y h](T)\rangle_\Omega$$

or, computing the adjoint,

$$\begin{aligned} \hat{F}'(y) = y + f_r w \qquad &\text{with} \quad -\dot{w} - \Delta w + \nabla(f_p w) - f_s w = 0 \\ w\,|_{\partial\Omega} = 0, \qquad &w(T) = z(T) - \zeta. \end{aligned} \tag{3.2}$$

This formal computation can be justified if z is sufficiently smooth. Unfortunately (except with a restriction on the dimension) $z(y)$ will *not* be smooth enough for this for arbitrary $y \in \mathfrak{Y}$. (For example, for the case $f(\cdot, r, s, p) = r - f_0(\cdot, s)$ (f_0 increasing in s) considered in [4] we needed $z[y]$ in $L^3(\mathscr{Q})$ at one stage of the justification of differentiability and this holds in $\mathfrak{Z}$ subject to the restriction $m < 10$.)

On the other hand, if we set

$$\hat{G}(\hat{y}, y) := \hat{y} + f_r w, \qquad \mathbf{H}(y) :- f_r w \tag{3.3}$$

with w given by (3.2) (f_r, f_s, f_p evaluated at $(\cdot, y, z[y], \nabla z[y])$], then we may be able to verify Hypotheses 3.1 and 3.2. Begin by noting the relgularity of f_r, f_s, f_p as evaluated at $(\cdot, y, z, \nabla z)$ with $y \in \mathfrak{Z}$, $z \in \mathfrak{Z}$, and the resulting regularity obtainable from (3.2) for w and so for $\mathbf{H}(y)$. This determines the choice of the space $\mathfrak{X}$ (which, for now, we assume is adequate to justify differentiability) and we obtain from (3.2) and (3.3) bounds on w in $\mathfrak{X}$ depending on bounds for y in $\mathfrak{Y}$ and the resulting bounds for z in $\mathfrak{Z}$. Since we have an a priori bound for y in $\mathscr{F}_a$ from the form of $\hat{F}$, this gives Hypothesis 3.2(i); in the present case Hypothesis 3.2(ii) is trivial as $\hat{G}(\mathbf{H}(y), y) \equiv 0$. For Hypothesis 3.1, note that $\hat{G}$ is linear in its first factor and continuous in both so $y_j \to^{\mathfrak{Y}} \bar{y}$ gives $\hat{G}(y_j, y_j) \to \hat{G}(\bar{y}, \bar{y})$ while $x_j \rightharpoonup \bar{x}$ (in $\mathfrak{X}_w$) then gives $\hat{G}(x_j, y_j) \to \hat{G}(\bar{x}, \bar{y})$. Thus, if these limits are each 0, we have $\bar{y} = \mathbf{H}(\bar{y})$ rom the first and $\bar{x} = \mathbf{H}(\bar{y})$ from the second so $\bar{x} = \bar{y}$.

If this construction does not give a strong enough space $\mathfrak{X}$ to justify the differentiability, we could proceed as follows: given y, define $y_1 = \mathbf{H}_1(y)$; using this (noting $y_1 \in \mathfrak{Y}_1$ smoother than y so $z[y_1]$ is in a space $\mathfrak{Z}_1$ smoother than $z[y]$), we repeat the process to get $y_2 = \mathbf{H}_1(y_1)$ and set $\mathbf{H}(y) = y_2$ and $\hat{G}(\hat{y}, y) = \hat{y} - y_2$. If this is inadequate, we can use additional iterations....)

It will be seen readily that this construction corresponds precisely to the argument (c) and (d) which was originally circular: the construction of $\mathbf{H}(\cdot)$ corresponds to the way in which we *would* have shown regularity of the optimal control $\bar{y}$ using the (formally obtained) necessary condition—represented by (3.2)—if we knew that (step (d)), while the justification of differentiability for the restriction to $\mathfrak{X}$ is as earlier (step (c)). *The contribution of Theorem* 3.3 *is that this argument is no longer circular.*

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